

# IMPROVEMENT OF GRAPH THEORY WEI'S INEQUALITY<sup>\*†</sup>

Nedyalko Dimov Nenov

February 16, 2008

## Abstract

Wei in [8] and [9] discovered a bound on the clique number of a given graph in terms of its degree sequence. In this note we give an improvement of this result.

We consider only finite non-oriented graphs without loops and multiple edges. A set of  $p$  vertices of a graph is called a  $p$ -clique if each two of them are adjacent. The greatest positive integer  $p$  for which  $G$  has a  $p$ -clique is called clique number of  $G$  and is denoted by  $\text{cl}(G)$ . A set of vertices of a graph is independent if the vertices are pairwise nonadjacent. The independence number  $\alpha(G)$  of a graph  $G$  is the cardinality of a largest independent set of  $G$ .

In this note we shall use the following notations:

- $V(G)$  is the vertex set of graph  $G$ ;
- $N(v)$ ,  $v \in V(G)$  is the set of all vertices of  $G$  adjacent to  $v$ ;
- $N(V)$ ,  $V \subseteq V(G)$  is the set  $\bigcap_{v \in V} N(v)$ ;
- $d(v)$ ,  $v \in V(G)$  is the degree of the vertex  $v$ , i.e.  $d(v) = |N(v)|$ .

Let  $G$  be a graph,  $|V(G)| = n$  and  $V \subseteq V(G)$ . We define

$$W(V) = \sum_{v \in V} \frac{1}{n - d(v)};$$

$$W(G) = W(V(G)).$$

Wei in [8] and [9] discovered the inequality

$$\alpha(G) \geq \sum_{v \in V(G)} \frac{1}{1 + d(v)}.$$

Applying this inequality to the complementary graph of  $G$  we see that it is equivalent to the following inequality

$$\text{cl}(G) \geq \sum_{v \in V(G)} \frac{1}{n - d(v)}$$

---

<sup>\*</sup>2000 Mathematics Subject Classification: 05C35

<sup>†</sup>Key words: clique number, degree sequence

that is

$$(1) \quad \text{cl}(G) \geq W(G).$$

Alon and Spencer [1] gave an elegant probabilistic proof of Wei's inequality. In the present note we shall improve the inequality (1).

**Definition 1.** Let  $G$  be a graph,  $|V(G)| = n$  and  $V \subseteq V(G)$ . The set  $V$  is called a  $\delta$ -set in  $G$ , if

$$d(v) \leq n - |V|$$

for all  $v \in V$ .

**Example 1.** Any independent set  $V$  of vertices of a graph  $G$  is a  $\delta$ -set in  $G$  since  $N(v) \subseteq V(G) \setminus V$  for all  $v \in V$ .

**Example 2.** Let  $V \subseteq V(G)$  and  $|V| \geq \max\{d(v), v \in V(G)\}$ . Since  $d(v) \leq |V|$  for all  $v \in V(G)$ ,  $V(G) \setminus V$  is a  $\delta$ -set in  $G$ .

The next statement obviously follows from Definition 1:

**Proposition 1.** Let  $V$  be a  $\delta$ -set in a graph  $G$ . Then  $W(V) \leq 1$ .

**Definition 2.** A graph  $G$  is called an  $r$ -partite graph if

$$V(G) = V_1 \cup \dots \cup V_r, \quad V_i \cap V_j = \emptyset, \quad i \neq j,$$

where the sets  $V_i$ ,  $i = 1, \dots, r$ , are independent. If the sets  $V_i$ ,  $i = 1, \dots, r$ , are  $\delta$ -sets in  $G$ , then  $G$  is called generalized  $r$ -partite graph. The smallest integer  $r$  such that  $G$  is a generalized  $r$ -partite graph is denoted by  $\varphi(G)$ .

**Proposition 2.**  $\varphi(G) \geq W(G)$ .

*Proof.* Let  $\varphi(G) = r$  and

$$V(G) = V_1 \cup \dots \cup V_r, \quad V_i \cap V_j = \emptyset, \quad i \neq j,$$

where  $V_i$ ,  $i = 1, \dots, r$ , are  $\delta$ -sets in  $G$ . Since  $V_i \cap V_j = \emptyset$ ,  $i \neq j$ , we have

$$W(G) = \sum_{i=1}^r W(V_i).$$

According to Proposition 1  $W(V_i) \leq 1$ ,  $i = 1, \dots, r$ . Thus  $W(G) \leq r = \varphi(G)$ .  $\square$

Below (see Theorem 1) we shall prove that  $\text{cl}(G) \geq \varphi(G)$ . Thus (1) follows from Proposition 2.

**Definition 3** ([2]). Let  $G$  be a graph and  $v_1, \dots, v_r \in V(G)$ . The sequence  $v_1, \dots, v_r$  is called an  $\alpha$ -sequence in  $G$  if the following conditions are satisfied:

- (i)  $d(v_1) = \max\{d(v) \mid v \in V(G)\}$ ;
- (ii)  $v_i \in N(v_1, \dots, v_{i-1})$  and  $v_i$  has maximal degree in the graph  $G[N(v_1, \dots, v_{i-1})]$ ,  $2 \leq i \leq r$ .

Every  $\alpha$ -sequence  $v_1, \dots, v_s$  in the graph  $G$  can be extended to an  $\alpha$ -sequence  $v_1, \dots, v_s, \dots, v_r$  such that  $N(v_1, \dots, v_{r-1})$  be a  $\delta$ -set in  $G$ . Indeed, if the  $\alpha$ -sequence  $v_1, \dots, v_s, \dots, v_r$  is such that it is not continued in a  $(r+1)$ -clique (i.e.  $v_1, \dots, v_s, \dots, v_r$  is a maximal  $\alpha$ -sequence in the sense of inclusion) then  $N(v_1, \dots, v_{r-1})$  is an independent set and, therefore, a  $\delta$ -set in  $G$ . However, there are  $\alpha$ -sequences  $v_1, \dots, v_r$  such that  $N(v_1, \dots, v_{r-1})$  is a  $\delta$ -set but it is not an independent set.

**Theorem 1.** *Let  $G$  be a graph and  $v_1, \dots, v_r$ ,  $r \geq 2$ , be an  $\alpha$ -sequence in  $G$  such that  $N(v_1, \dots, v_{r-1})$  is a  $\delta$ -set in  $G$ . Then*

- (a)  $\varphi(G) \leq r \leq \text{cl}(G)$ ;
- (b)  $r \geq W(G)$ .

*Proof.* According to Definition 3  $v_1, \dots, v_r$  is an  $r$ -clique and thus  $r \leq \text{cl}(G)$ . Since  $N(v_1, \dots, v_{r-1})$  is a  $\delta$ -set, the graph  $G$  is a generalized  $r$ -partite graph, [6]. Hence  $r \geq \varphi(G)$ . The inequality (b) follows from (a) and Proposition 2.  $\square$

**Remark.** Theorem 1 (b) was proved in [7] in the special case when  $N(v_1, \dots, v_{r-1})$  is independent set in  $G$ .

**Definition 4.** *Let  $G$  be a graph and  $v_1, \dots, v_r \in V(G)$ . The sequence  $v_1, \dots, v_r$  is called  $\beta$ -sequence in  $G$  if the following conditions are satisfied:*

- (i)  $d(v_1) = \max\{d(v) \mid v \in V(G)\}$ ;
- (ii)  $v_i \in N(v_1, \dots, v_{i-1})$  and  $d(v_i) = \max\{d(v) \mid v \in N(v_1, \dots, v_{r-1})\}$ ,  $2 \leq i \leq r$ .

**Theorem 2.** *Let  $v_1, \dots, v_r$  be a  $\beta$ -sequence in a graph  $G$  such that*

$$d(v_1) + \dots + d(v_r) \leq (r-1)n,$$

*where  $n = |V(G)|$ . Then  $r \geq W(G)$ .*

*Proof.* According to [5] it follows from  $d(v_1) + \dots + d(v_r) \leq (r-1)n$  that  $G$  is a generalized  $r$ -partite graph. Hence  $r \geq \varphi(G)$  and Theorem 2 follows from Proposition 2.  $\square$

**Corollary 1.** *Let  $G$  be a graph,  $|V(G)| = n$  and  $v_1, \dots, v_r$  be a  $\beta$ -sequence in  $G$  which is not contained in  $(r+1)$ -clique. Then  $r \geq W(G)$ .*

*Proof.* Since  $v_1, \dots, v_r$  is not contained in  $(r+1)$ -clique it follows that  $d(v_1) + \dots + d(v_r) \leq (r-1)n$ , [3].  $\square$

**Theorem 3.** *Let  $G$  be a graph,  $|V(G)| = n$  and  $v_1, \dots, v_r$ ,  $r \geq 2$ , be a  $\beta$ -sequence in  $G$  such that  $N(v_1, \dots, v_{r-1})$  is a  $\delta$ -set in  $G$ . Then  $r \geq W(G)$ .*

*Proof.* Since  $N(v_1, \dots, v_{r-1})$  is a  $\delta$ -set according to [6] there exists an  $r$ -partition

$$V(G) = V_1 \cup \dots \cup V_r, \quad V_i \cap V_j = \emptyset, \quad i \neq j,$$

where  $V_i$ ,  $i = 1, \dots, r$ , are  $\delta$ -sets and  $v_i \in V_i$ . Thus, we have

$$d(v_i) \leq n - |V_i|, \quad i = 1, \dots, r.$$

Summing up these inequalities we obtain that  $d(v_1) + \dots + d(v_r) \leq (r-1)n$ . Therefore Theorem 3 follows from Theorem 2.  $\square$

## References

- [1] N. ALON, J. H. SPENCER. The Probabilistic Method. Wiley, New York, 1992.
- [2] N. KHADZHIIVANOV, N. NENOV. Extremal problems for  $s$ -graphs and a theorem of Turan. *Serdica*, 3 (1977), 117–125 (in Russian).
- [3] N. KHADZHIIVANOV, N. NENOV. Sequences of maximal degree vertices in graphs. *Serdica Math. J.*, 30 (2004), 95–102.
- [4] N. KHADZHIIVANOV, N. NENOV. Generalized  $r$ -partite graphs and Turan's Theorem. *Compt. rend. Acad. bulg. Sci.*, 57, No 2 (2004), 19–24.
- [5] N. KHADZHIIVANOV, N. NENOV. Saturated  $\beta$ -sequences in graphs. *Compt. rend. Acad. bulg. Sci.*, 57, No 6 (2004), 49–54.
- [6] N. KHADZHIIVANOV, N. NENOV. Balanced vertex sets in graph. *Ann. Univ. Sofia, Fac. Math. Inf.*, 97 (2005), 50–64.
- [7] O. MURPHY. Lower bounds on the stability number of graph computed in terms of degree. *Discrete Math.*, 90 (1991), 207–211.
- [8] V. K. WEI. A lower bound on the stability number of a simple graph. Bell Laboratories Technical Memorandum, 81–11217–9. Murray Hill, NJ, 1981.
- [9] V. K. WEI. Coding for a Multiple Access Channel. Ph. D. Thesis, University of Hawaii, Honolulu, 1980.

Nedyalko Dimov Nenov  
Faculty of Mathematics and Informatics  
St Kliment Ofridski University of Sofia  
5, James Bourchier Blvd.  
BG-1164 Sofia, Bulgaria  
e-mail: [nenov@fmi.uni-sofia.bg](mailto:nenov@fmi.uni-sofia.bg)